Exercise 31

Find the exact value of $\int_C x^3y^2z\,ds$, where C is the curve with parametric equations $x = e^{-t}\cos 4t$, $y = e^{-t}\sin 4t$, $z = e^{-t}$, $0 \le t \le 2\pi$.

Solution

With this parameterization in t, the line integral becomes

$$\begin{aligned}
\int_C x^3y^2z\,ds &= \int_0^{2\pi} [x(t)]^3[y(t)]^2 z(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt \\
&= \int_0^{2\pi} (e^{-t}\cos 4t)^3(e^{-t}\sin 4t)^2(e^{-t}) \\
&\qquad \times \sqrt{(-e^{-t}\cos 4t - 4e^{-t}\sin 4t)^2 + (-e^{-t}\sin 4t + 4e^{-t}\cos 4t)^2 + (-e^{-t})^2}\,dt \\
&= \int_0^{2\pi} (e^{-3t}\cos^3 4t)(e^{-2t}\sin^2 4t)(e^{-t}) \\
&\qquad \times \sqrt{(-e^{-t}\cos 4t - 4e^{-t}\sin 4t)^2 + (-e^{-t}\sin 4t + 4e^{-t}\cos 4t)^2 + (-e^{-t})^2}\,dt \\
&= \int_0^{2\pi} (e^{-6t}\cos^3 4t\sin^2 4t)\sqrt{18e^{-2t}}\,dt \\
&= \int_0^{2\pi} (\sqrt{18}e^{-7t}\cos^3 4t\sin^2 4t)\,dt.
\end{aligned}$$

Plugging this integral into Mathematica yields the following result.

$$\int_C x^3y^2z\,ds = \frac{172\,704}{5\,632\,705}\sqrt{2}(1 - e^{-14\pi}) \approx 0.0433611$$